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Renormalization group invariant average momentum of non-singlet parton densities

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Abstract

We compute, within the Schrödinger functional scheme, a renormalization group invariant renormalization constant for the first moment of the non-singlet parton distribution function. The matching of the results of our non-perturbative calculation with the ones from hadronic matrix elements allows us to obtain eventually a renormalization group invariant average momentum of non-singlet parton densities, which can be translated into a preferred scheme at a specific scale.

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Physical quantities that need renormalization, such as the coupling constant, the quark mass or the matrix elements of operators appearing in the Wilson operator product expansion with a non-zero anomalous dimension, are “running” with the renormalization scale. The choice of the scale is in general motivated by the kinematics of the Green functions involving such renormalized quantities, but the final physical predictions of the theory without perturbative approximations are independent of such a choice. This leads to the well-known renormalization group equations that put the independence on a formal basis. The redundancy in a parametrisation of the theory in terms of renormalized quantities and the relative renormalization scale can be avoided by considering renormalization group invariant quantities, such as the Λ parameter of QCD or the renormalization group invariant quark masses (RGIM). The advantage of the latter choice in non-perturbative lattice determinations of the quark mass has recently been stressed by the authors of ref. [1] where an essential part of the RGIM programme was carried out.

In particular, the definition of the RGIM, which corresponds, roughly speaking, to a running mass at infinite renormalization scale, is free of the renormalization scheme dependence that usually affects quantities renormalized (in a given scheme) at a fixed scale. It can hence be evolved back to an arbitrary finite scale in a preferred scheme.

The purpose of this letter is to present a similar calculation for the operator that corresponds to the average momentum of non-singlet parton densities. A lattice — perturbative and non-perturbative — study of the evolution of such an operator has been discussed in refs. [2] and [3] to which we address the reader for more details about the calculation that we here only shortly summarize as follows. We calculate the renormalization constant of the twist-two non-singlet operator for the first moment of the quark parton distribution defined by:

$$\mathcal{O}_{\mu\nu}^{qNS} = \left(\frac{i}{2}\right)^{n-1} \bar{\psi}(x) \gamma_{\{\mu} \overleftrightarrow{D}_{\nu\}} \frac{\lambda^f}{2} \psi(x) - \text{trace terms} , \quad (1)$$

where $\{\dots\}$ means symmetrization of the indices. We remark that the technique discussed here can be extended to higher moments in an analogous way. The basic ingredient for the reconstruction of the non-perturbative scale dependence of the renormalization constants of the above operator is the finite-size step scaling

function σ_Z defined by:

$$Z(sL) = \sigma_Z(s, \bar{g}^2(L))Z(L) , \quad (2)$$

where L is the physical length that plays the role of the renormalization scale, s parametrizes the step size of the change in the scale, and Z is the renormalization constant of the operator, which is defined by:

$$\mathcal{O}^R(\mu) = Z(1/\mu)^{-1} \mathcal{O}^{\text{bare}}(1/L) . \quad (3)$$

Z is obtained from the Schrödinger Functional (SF) matrix element, $\langle \dots \rangle_{\text{SF}}$, of the operator on a finite volume L^3T , normalized to its tree level¹:

$$\langle \mathcal{O}^{\text{bare}}(1/L) \rangle_{\text{SF}} = Z(L) \langle \mathcal{O}^{\text{tree}} \rangle_{\text{SF}} . \quad (4)$$

The renormalized operator then satisfies $\langle \mathcal{O}^R(\mu = 1/L) \rangle_{\text{SF}} = \langle \mathcal{O}^{\text{tree}} \rangle_{\text{SF}}$. The framework of the Schrödinger Functional [4, 5], which describes the quantum time evolution between two fixed classical gauge and fermion configurations, defined at times $t = 0$ and $t = T$, has been used extensively in the recent literature [6, 7, 1] to calculate non-perturbative renormalization constants of local operators. Among the advantages of the method, we only quote the possibility of performing the computations at zero physical quark mass and of using non-local gauge-invariant sources for the fermions without need of a gauge-fixing procedure. In our particular case, we exploit both features. Our observable is defined by [2]:

$$Z = \frac{f_2(x_0 = L/4)}{\sqrt{f_1}} \bigg/ \left(\frac{f_2(x_0 = L/4)}{\sqrt{f_1}} \right)_{\text{tree}} , \quad (5)$$

with f_2 given by

$$f_2(x_0) = -a^6 \sum_{\mathbf{y}, \mathbf{z}} e^{i\mathbf{p}(\mathbf{y}-\mathbf{z})} \langle \frac{1}{4} \bar{\psi}(\mathbf{x}) \gamma_{\{1} \overleftrightarrow{D}_{2\}} \frac{1}{2} \tau^3 \psi(\mathbf{x}) \bar{\zeta}(\mathbf{y}) \Gamma \frac{1}{2} \tau^3 \zeta(\mathbf{z}) \rangle \quad (6)$$

and f_1 by

$$f_1 = -a^{12} \sum_{\mathbf{y}, \mathbf{z}, \mathbf{v}, \mathbf{w}} \langle \bar{\zeta}'(\mathbf{v}) \frac{\tau^3}{2} \zeta'(\mathbf{w}) \bar{\zeta}(\mathbf{x}) \frac{\tau^3}{2} \zeta(\mathbf{y}) \rangle , \quad (7)$$

¹In the following we choose $T = L$.

where $\zeta = \delta/\delta\bar{\psi}_c$ and $\bar{\zeta} = -\delta/\delta\psi_c$ are the derivatives with respect to the two-component classical fermion fields ($\bar{\psi}_c$ and ψ_c , respectively) at the boundary $x_0 = 0$, while ζ' and $\bar{\zeta}'$ are the corresponding derivatives at the boundary $x_0 = T$. The projection on the classical components is achieved by the projector P_{\pm} defined by $\frac{1}{2}(1 \pm \gamma_0)$. On the boundaries, the theory possesses only a *global* gauge invariance that is preserved by the quantities defined above. The values of x_0 (set to $T/4$) and of the non-zero component of the momentum p_x (set to $2\pi/L$) are both scaled in units of L , which therefore remains the only scale besides the lattice spacing a . The quantity f_1 serves as a normalization factor that removes the wave function renormalization constant of the ζ fields in order to isolate the running associated with the operator in eq. (1) only.

The determination of the step scaling function in the continuum has been shown to be universal with respect to the lattice action used in ref. [8]. From a fit to its dependence upon the running coupling constant \bar{g}^2 , renormalized in the SF scheme, we can extract the following “running” step scaling function:

$$\sigma(\mu/\mu_0, \bar{g}^2(\mu_0)) = Z(1/\mu)/Z(1/\mu_0) \quad (8)$$

i.e. the renormalization constant normalized to the one at a reference scale μ_0 . The running operator matrix element at the scale μ , which we denote generically by the symbol O , can be defined in terms of the one at scale μ_0 simply by:

$$O^{ren}(\mu) = O^{ren}(\mu_0)\sigma(\mu/\mu_0, \bar{g}^2(\mu_0)) . \quad (9)$$

The scale dependence of the renormalized operator just reflects the one of its renormalization constant governed by the equation:

$$\frac{dZ(1/\mu)}{d\log(\mu)} = Z(1/\mu) \cdot \gamma_O(g^2(\mu)) , \quad (10)$$

from which follows:

$$\frac{dO^{ren}(\mu)}{d\log(\mu)} = O^{ren}(\mu) \cdot \gamma_O(g^2(\mu)) . \quad (11)$$

Following ref. [1] but using a slightly different normalization in taking out the factor of $2b_0$, we define, for operators entering the Wilson operator product expansion, a renormalization group invariant matrix element:

$$O_{INV}^{ren} = O^{ren}(\mu) \cdot (\bar{g}^2)^{-\gamma_0/2b_0} \exp \left\{ - \int_0^{\bar{g}} dg \left[\frac{\gamma(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g} \right] \right\} , \quad (12)$$

where for the anomalous dimension function $\gamma(g)$ and the β -function the expressions up to three loops may be inserted for values of g small enough that perturbation theory can be trusted:

$$\gamma(g^2(\mu)) = \gamma_0 g^2(\mu) + \gamma_1 g^4(\mu) + \gamma_2 g^6(\mu), \quad (13)$$

$$\beta(g^2(\mu)) = \beta_0 g^4(\mu) + \beta_1 g^6(\mu) + \beta_2 g^8(\mu). \quad (14)$$

We note that for $\gamma(g)$ we know the effective three-loop term from our non-perturbative computation of γ_2 [3], while γ_0 and γ_1 are given from perturbation theory.

From eq. (8) once the $O^{ren}(\mu_0)$ is known in some scheme, for example the SF scheme we have described, we can obtain the renormalization scheme invariant matrix element by introducing an “ultraviolet invariant” running step scaling function ² defined by:

$$\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0) = \sigma(\mu/\mu_0, \bar{g}^2(\mu_0)) \cdot (\bar{g}^2(\mu))^{-\gamma_0/2b_0} \exp \left\{ - \int_0^{\bar{g}(\mu)} dg \left[\frac{\gamma(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g} \right] \right\} \quad (15)$$

as follows:

$$O_{\text{INV}}^{ren} = O^{ren}(\mu_0) \cdot \mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0). \quad (16)$$

The scale μ_0 is in general a low-energy scale, where the hadronic matrix element can be calculated without severe finite volume effects. In our case, it can be identified with a low-energy scale at which the evolution of the renormalization constant can be started. In particular we shall fix this scale to be $2L_{\text{max}}$ as in ref. [1]. Recently, L_{max} has been computed in terms of the low energy reference quantity r_0 [9] in [10]. In order to “step down” from this scale, we will need the step scaling function with $s = 2$, i.e. starting from $\bar{g}^2(L_{\text{max}}) = 3.48$, our largest value of \bar{g}^2 , we evolve with a step size of 2 until contact with perturbation theory can be made.

In this paper, we calculate, as a first step towards the computation of the renormalization group invariant matrix element, the quantity $\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0 = \frac{1}{2L_{\text{max}}})$. Note

²We remark that the invariance holds with respect to a change of the “ultraviolet” scale μ and not of the “infrared” scale μ_0 .

that $\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$ still depends on the reference scale μ_0 . The dependence on μ_0 will only disappear later, when it will be matched with the proper hadronic matrix element, making $O_{\text{INV}}^{\text{ren}}$ renormalization scheme independent.

In order to rely on the perturbative expansion for the β - and γ -functions appearing in eq. (15), we had to extend the calculation of our non-perturbative running to higher scales. We added four more values of \bar{g} for the step scaling function that now covers, in total, values of $\bar{g}^2(L)$ ranging from $\bar{g}^2(L) = 3.48$ to $\bar{g}^2(L) = 0.8873$. For the results at the four lowest values of \bar{g} we used the non-perturbatively improved clover action [11]; in figs. 1 and 2 we report the continuum extrapolation, for the values of \bar{g}^2 not presented already in ref. [8] of the step scaling function of the quantities f_1 and f_2 of eq. (5) ($\sigma_{\bar{Z}}$ and σ_{f_1} , respectively, see [3]): at smaller length scales the effects of lattice artefacts for $\sigma_{\bar{Z}}$ are progressively reduced and the extrapolations become flatter.

From the results for σ_Z at the, in total, nine values of \bar{g} , we can make a fit to the step scaling function as a function of $\bar{g}^2(L)$. The results for σ_Z at the five largest values of $\bar{g}^2(L)$ are taken from the combined data presented in [8]. In ref. [3] we have shown that, in the scheme we adopted, the coefficient of the two-loop anomalous dimension is very large, when compared for example to the one in the $\overline{\text{MS}}$ scheme. We have also shown that this coefficient reduces by changing the expansion parameter, i.e. by using $\bar{g}^2(L/4)$ instead of $\bar{g}^2(L)$. The step scaling function as a function of $\bar{g}^2(L/4)$ is well fitted numerically by a polynomial in $\bar{g}^2(L/4)$ of the form:

$$\sigma(\bar{g}^2(L/4)) = 1 - \gamma_0 \log(2) \bar{g}^2 + c_4 \cdot \bar{g}^4 + c_6 \cdot \bar{g}^6 + c_8 \cdot \bar{g}^8, \quad (17)$$

where $\gamma_0 = 4/(9\pi^2)$. The final results stay unchanged when we switch to a two-parameter fit that also gives a very good χ^2 . We show our data for σ_Z as a function of $\bar{g}^2(L/4)$ together with the fit of eq. (17) in fig. 3.

From this fit we can construct the running step scaling function of eq. (15) with $\mu_0 = (2L_{\text{max}})^{-1}$. The result is shown in fig. 4, where we have used the two-loop expression for $\gamma(g)$ and the 3-loop expression for $\beta(g)$. By using eq. (15) we can finally estimate the value of $\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$: in the second column of table 1 we report the values of $\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$ as a function of the scale μ : for large values of μ the function, within the errors, approaches a plateau. We make a fit to a constant

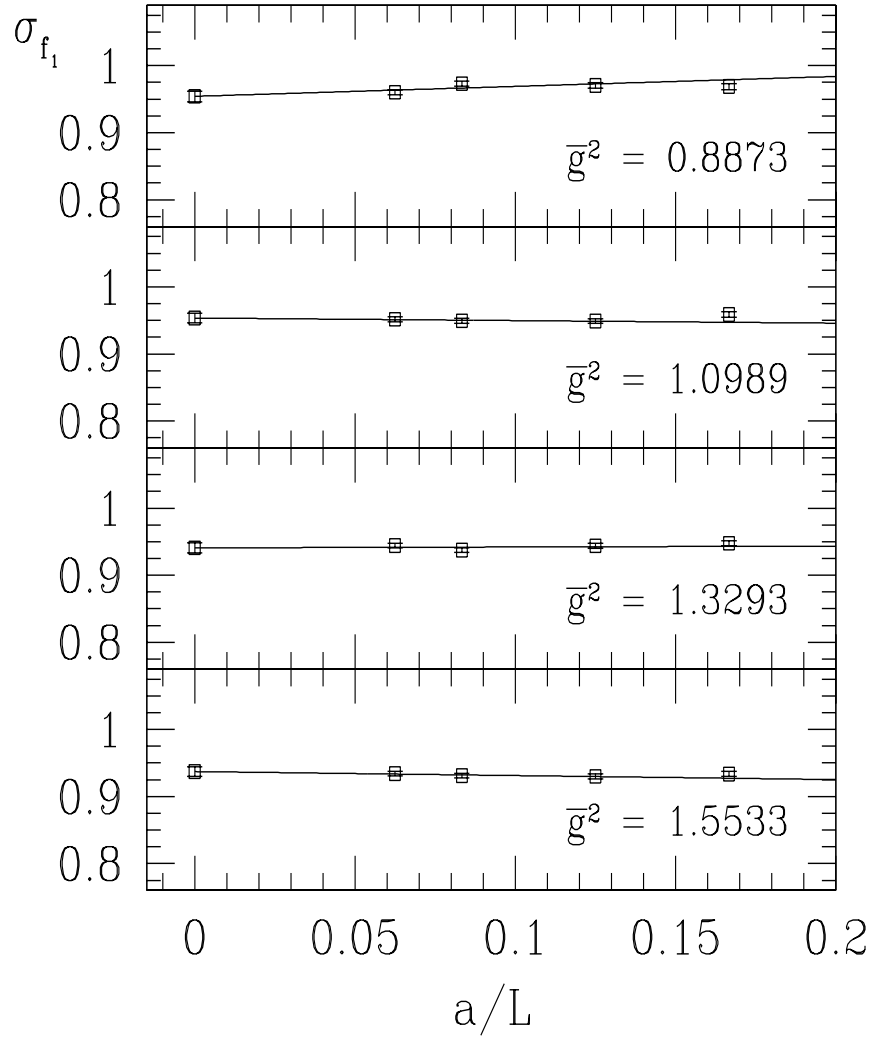


Figure 1: Continuum extrapolation of σ_{f_1} using a linear fit to the three data points with smallest values of a/L for the most perturbative values of \bar{g}^2 we have used in our work, which are indicated in the figure.

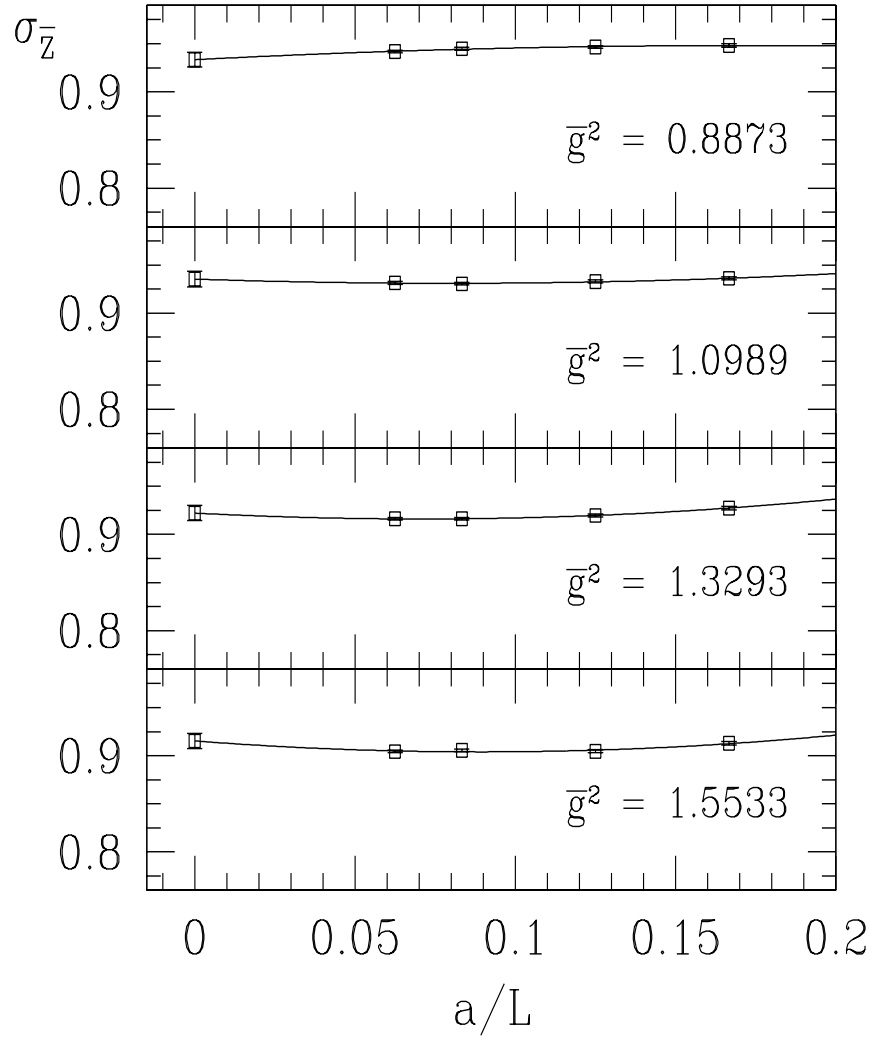


Figure 2: Continuum extrapolation of σ_Z using a quadratic fit to all four data points for the most perturbative values of \bar{g}^2 we have used in our work, which are indicated in the figure.

μ/μ_0	$\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$ $\bar{g}(L/4)$ γ_O 2-loop	$\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$ $\bar{g}(L)$ γ_O 2-loop	$\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$ $\bar{g}(L/4)$ γ_O 3-loop	$\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$ $\bar{g}(L)$ γ_O 3-loop
2^1	1.09(1)	1.33(1)	1.16(1)	1.18(1)
2^2	1.10(2)	1.24(2)	1.15(2)	1.16(2)
2^3	1.11(2)	1.20(2)	1.14(2)	1.15(2)
2^4	1.11(3)	1.18(2)	1.14(3)	1.14(2)
2^5	1.11(3)	1.16(3)	1.13(3)	1.14(3)
2^6	1.11(4)	1.15(3)	1.13(4)	1.13(3)
2^7	1.11(4)	1.14(3)	1.12(4)	1.13(3)
2^8	1.10(5)	1.14(3)	1.12(5)	1.13(3)
2^9	1.10(5)	1.13(3)	1.11(5)	1.12(3)

Table 1: The values for $\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$ when different scales μ are taken for matching with perturbation theory.

for the results ranging from $\mu/\mu_0 = 2^5$ to $\mu/\mu_0 = 2^9$, and we finally quote:

$$\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0 = (2L_{\text{max}})^{-1}) = 1.11(2) . \quad (18)$$

The invariant step scaling function is still scheme-dependent, because of the presence of the reference scale μ_0 . This will be cancelled only in the combination that defines the invariant matrix element. However, at fixed μ_0 , it should be independent of the choice of $\bar{g}^2(L/4)$ or of $\bar{g}^2(L)$ in the fit to the step scaling function. We therefore repeated the whole procedure described above by fitting the step scaling function as a function of $\bar{g}^2(L)$ and by using the correspondingly modified gamma function to two loops. The results are given in the third column of table 1. They are fully compatible with those obtained from the case $L/4$, although the plateau starts at higher energies, as expected. We report the comparison of both cases also in fig. 4. In the fourth and fifth column of table 1 we report the result for the case “ $L/4$ ” and “ L ” respectively, after including our estimate of the three-loop anomalous dimensions for the two cases, determined in [3, 8]. Not surprisingly, the two cases get close to each other more precociously. An estimate of the renormalization group invariant yields $\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0 = \frac{1}{2L_{\text{max}}}) = 1.14(2)$, again consistent with our earlier results using $\bar{g}(L/4)$ as expansion parameter.

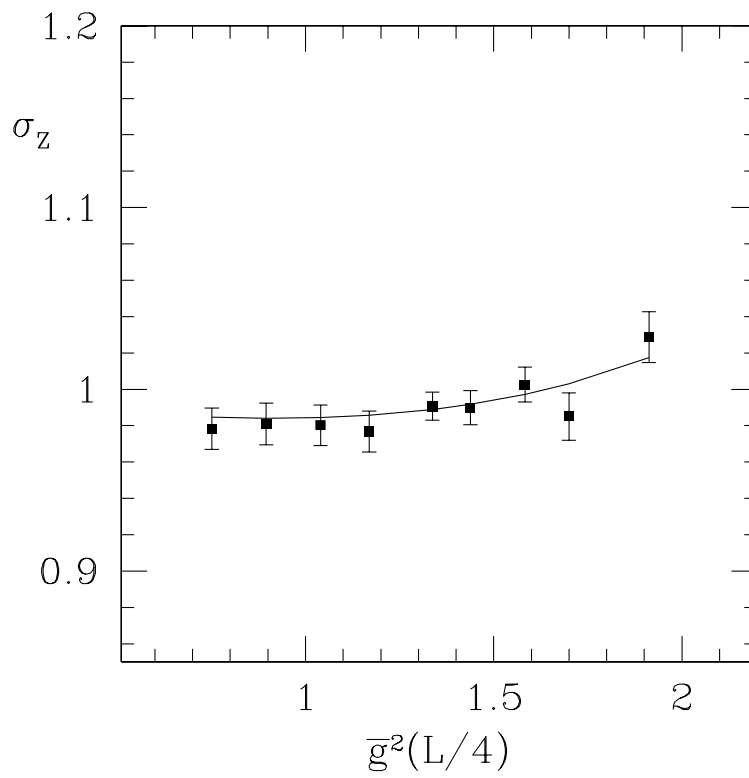


Figure 3: Our fit to the step scaling function.

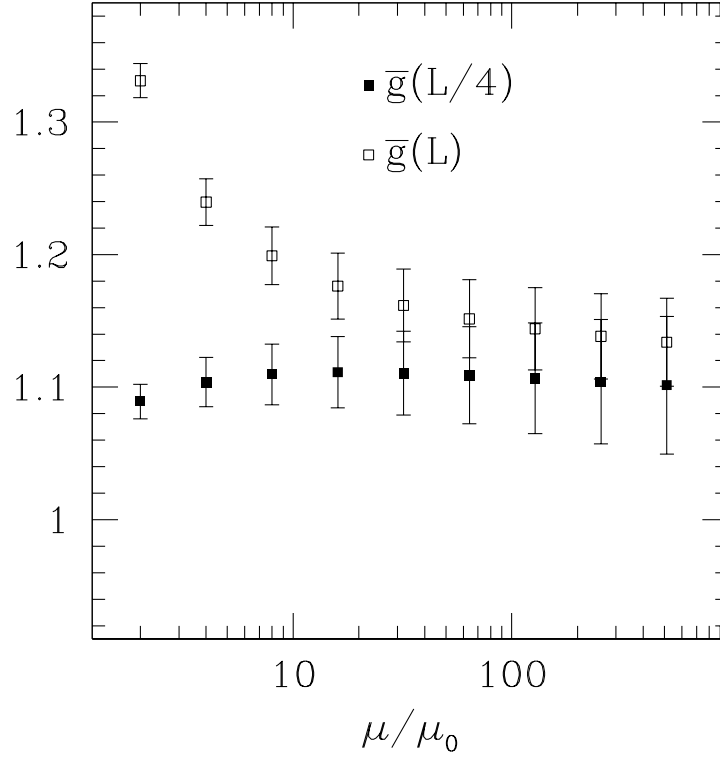


Figure 4: $\mathfrak{S}_{\text{INV}}^{\text{UV}}(\mu_0)$ as a function of the scale μ normalized to our reference scale $\mu_0 = (2L_{\text{max}})^{-1}$.

Matching the results of this paper with a non-perturbative calculation of the hadronic matrix element, in the continuum, in the same scheme and at the same reference energy scale, leads to the definition of a renormalization group invariant matrix element that can be confronted with experiment at any scale and in a preferred scheme. Such a calculation is in progress.

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